OD-matrix Estimation Based on a Dual Formulation of Traffic Assignment Problem*

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Congestion, accidents, greenhouse gas emission and others seem to become unsolvable problems for all levels of management in modern large cities worldwide. The increasing motorization dynamics requires development of innovative methodological tools and technical devices to cope with problems emerging on the road networks. First of all, control system for urban traffic area has to be created to support decision makers by processing a big volume of transportation data. The input for such a system is a volume of travel demand between origins and destinations — OD-matrix. The present work is devoted to the problem of OD-matrix estimation. Original OD-matrix estimation technique is offered by virtue of plate scanning sensors location. Mathematically developed technique is based on a dual formulation of the traffic assignment problem (equal journey time by alternative routes between any OD-pair). Traffic demand between certain OD-pair is estimated due to journey time obtained from plate scanning sensors. Moreover, the explicit relationship between traffic demand and journey time is obtained for network of parallel routes with one OD-pair. Eventually, the developed method was experimentally implemented to the Saint-Petersburg road network.

Povzetek: OD-matrika povezuje izvore in cilje prometnih povezav. Prispevek se ukvarja z iskanjem približkov OD-matrike z dvojno formulacijo.

1   Introduction

OD-matrix estimation and reconstruction are urgent and complicated challenges, since road networks of modern cities are extremely large and intricate. In general, OD-matrix estimation and reconstruction are different problems: the first means to obtain approximate values, while the second has a goal to obtain precise values of actual traffic demands [1]. One of the first mathematical models for OD-matrix estimation was formulated in a form of bi-level program [2]. Despite numerous publications, this problem still attracts researchers from all over the world [3–8]. A detailed comparative analysis of methods for trip matrix estimation was made in [4]. From a practical perspective, the most promising technique is combination of data obtained both from plate scanning sensors and link-flow counts [5].

This paper is also devoted to OD-matrix estimation problem. We believe that a plate scanning sensor is highly efficient engineering equipment. Indeed, due to link-flow counts one could obtain solely amount of vehicles on the link, while plate scanning allows to estimate the average travel time between origin and destination by identification the vehicle in origin and destination points. Since travel time between an origin-destination pair is a Lagrange multiplier for a primal traffic assignment problem (TAP), it is the variable in a dual formulation of TAP. Therefore, we are able to formulate a new bi-level optimization program for OD-matrix estimation based on data obtained from link-flow plate scanning sensors on congested networks.

The present article is organized as follows. In Section 2 the network of parallel routes with one OD-pair is investigated. The idea of OD-matrix estimation based on information about travel times between OD-pairs is clarified. Section 3 provides a dual formulation of traffic assignment problem in two subsections: for a simple network with one OD-pair and parallel routes, and for the general topology network. Section 4 describes bi-level optimization program for OD-matrix estimation on a congested network by virtue of plate scanning sensors. Section 5 is devoted to the experimental implementation of the developed approach to the Saint-Petersburg road network. Conclusions are given in Section 6.

2   The network of parallel routes

Let us ntroduce the following notation: F is traffic demand between OD-pair; f_i is traffic flow on the route i, $i = \overline{1, n}$, $f = (f_1, \ldots, f_n)$, $\sum_{i=1}^{n} f_i = F$; $t_i(f_i) = a_i + b_i f_i$ is

travel time on congested arc i, $i = \overline{1,n}$. In the present work we model travel time on a congested arc as the linear function.

Let us formulate traffic assignment problem on network of parallel routes as an optimization program [9, 10]:

$$z(f^*) = \min_f z(f) = \min_f \sum_{i=1}^n \int_0^{f_i} t_i(u)du, \quad (1)$$

subject to

$$\sum_{i=1}^n f_i = F, \quad (2)$$

$$f_i \geq 0 \quad \forall i = \overline{1,n}. \quad (3)$$

Wardrop's first principle states that the journey times in all routes actually used are equal and less than those that would be experienced by a single vehicle on any unused route [10, 11]. Traffic flows that satisfy this principle are usually referred to as "user equilibrium" (UE) flows, since each user chooses the route that is the best. On the network of parallel routes UE is reached by such assignment $f^* = (f_1^*, \ldots, f_n^*)$ as:

$$\begin{cases} t_i(f_i^*) = t^* > 0 & \text{when } f_i^* > 0, \\ t_i(f_i^*) > t^* & \text{when } f_i^* = 0, \end{cases} \quad i = \overline{1,n}.$$

Thus, the mathematically formalized idea of UE (1)–(3) can be used in reconstruction of traffic assignment on the network between origin-destination pair. On the other hand, if we know travel time t^* between OD-pair, we are able to reconstruct traffic demand F on the linear network of parallel routes.

Without loss of generality we assume that routes are numbered as follows:

$$a_1 \leq \ldots \leq a_n.$$

Theorem 1. *Traffic demand F for a linear network of parallel routes can be obtained explicitly:*

$$F = t^* \sum_{s=1}^k \frac{1}{b_s} - \sum_{s=1}^k \frac{a_s}{b_s}, \quad (4)$$

where k satisfies

$$a_1 \leq \ldots a_k < t^* \leq a_{k+1} \ldots \leq a_n. \quad (5)$$

Proof. Travel time t^* on used routes is the Lagrangian multiplier that corresponds to the restriction (2) of optimization program (1)–(3) [9, 12, 13].

Since goal function (1) is convex then the Kuhn-Tucker conditions are both necessary and sufficient. Let us introduce Lagrange multiplier μ for the flow conservation constraint (2) and multipliers $\eta_i \geq 0$, $i = \overline{1,n}$ for (3). The Lagrangian for optimization problem (1)–(3) is

$$L = \sum_{i=1}^n \int_0^{f_i} t_i(u)du + \mu\left(F - \sum_{i=1}^n f_i\right) + \\ + \sum_i \eta_i(-f_i). \quad (6)$$

The derivative of Lagrangian (6) at variable f_i has to be equal to zero:

$$\mu = t_i(f_i) - \eta_i.$$

Complementary slackness condition states that $\eta_i \cdot f_i = 0$. This equation holds when at least one of the variables is zero. Thus, if $f_i > 0$, then $\eta_i = 0$ and

$$\mu = t_i(f_i) = a_i + b_i f_i. \quad (7)$$

However, if $f_i = 0$, then $\eta_i \geq 0$ and

$$\mu = t_i(f_i) - \eta_i = a_i - \eta_i.$$

Hence, if $a_i \geq \mu$ then $f_i = 0$. On the contrary, if we express f_i in terms of μ from (7) we get

$$f_i = \frac{\mu - a_i}{b_i}.$$

Therefore, if $f_i > 0$ then

$$\mu > a_i.$$

Thus we are able to define the set of actually used routes (routes with nonzero flows):

$$f_i = \begin{cases} \frac{\mu - a_i}{b_i} & \text{when } a_i < \mu, \\ 0, & \text{when when } a_i \geq \mu, \end{cases} \quad i = \overline{1,n}. \quad (8)$$

Without loss of generality, we could renumber routes in such a way that $a_1 \leq \ldots \leq a_k < \mu \leq a_{k+1} \leq \ldots \leq a_n$. Then, due to (2) and (8) we obtain

$$\sum_{i=1}^n f_i = \sum_{s=1}^k \frac{\mu - a_s}{b_s} = F$$

and, consequently,

$$F = \sum_{s=1}^k \frac{\mu - a_s}{b_s}.$$

Eventually, since μ is t^* by definition, the theorem is proved. $\square$

Therefore, if we know travel time of a vehicle on any alternative route between OD-pair, appropriate traffic demand can be uniquely reconstructed. Due to such results the developed approach seems to be promising. The main idea of this method is based on the first principle of Wardrop is as follows: if we define journey time of a vehicle on any actually used route between certain OD-pair, then we believe that journey time on all other used routes is the same.
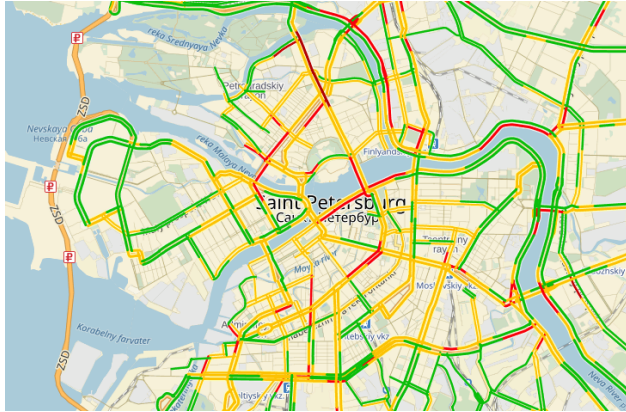
Figure 1: The traffic in Saint-Petersburg city

3 Dual formulation of TAP

Generally, t^* could be easily determined between any pair of origin and destination on a real city road network. Indeed, there are online services collecting information about average travel speeds on all arcs of a city road network.

For instance, "Yandex.Traffic" based on "Yandex.Maps" reflects the current road situation online (fig.1). Due to the large scale databases this service is able to make statistical predictions for different time periods and different days of week. Average travel speed through any arc of road network is in the public domain (fig.2).
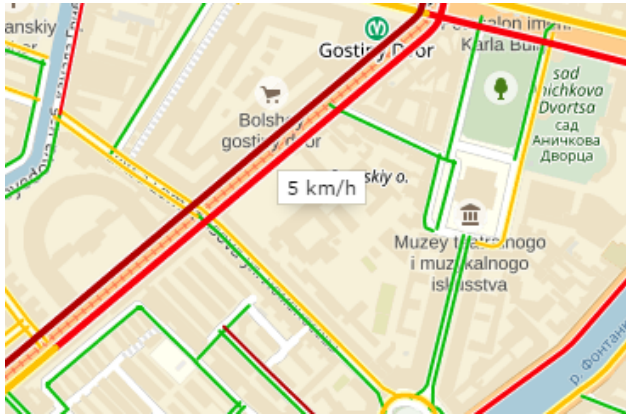

Figure 2: Data from Yandex.Maps

Therefore, we can easily determine travel time through the whole route between any pair of origin and destination, using the information about average speed on the arcs. We believe that road traffic is assigned according to the first Wardrop principle. Thereby, if we estimate travel time through the shortest route between OD-pair then we obtain t^* for this OD-pair. Note, that such information about road network could also be useful for more efficient allocation of resources by different companies [14].

Actually, due to information about equilibrium travel time between OD-pairs we can estimate OD-matrix. Indeed, let us refer to the duality theory of mathematical pro-

gramming.

3.1 Dual formulation of TAP for the network of parallel routes

We introduce dual variables $\eta = (\eta_1, \ldots, \eta_n)$ and define dual traffic equilibrium problem for the network of parallel routes:

$$\max_{\eta} \sum_{i=1}^{n} \theta_i(\eta),$$

with constraints

$$\eta_i \geq 0, \quad \forall i = \overline{1, n},$$

where $\theta_i(\eta)$ for all $i = \overline{1, n}$ satisfies the following optimization program

$$\theta_i(\eta) = \sum_{i=1}^{n} \min_{f_i} \left(\int_0^{f_i} t_i(u) du - \eta_i f_i \right),$$

subject to

$$\sum f_i = F.$$

It is proved that equilibrium assignment problem for a network of parallel routes can be reduced to the fixed point problem and is expressed explicitly [15]. Now let us prove that the above mentioned bi-level program is really dual TAP.

3.2 Dual formulation of TAP for a network of general topology

Let us consider a network of general topology presented by oriented graph $G = (N, A)$. We introduce the following notation: W is set of OD-pairs, $w \in W$, $W \in N$; K^w is the set of routes connecting OD-pair w; F^w is traffic demand for OD-pair w, $F = \left(F^1, \ldots, F^{|W|} \right)^{\mathrm{T}}$; f_k^w is traffic flow on the route $k \in K^w$, $\sum_{k \in K^w} f_k^w = F^w$; $f^w = \{f_k^w\}_{k \in K^w}$ and $f = \{f^w\}_{w \in W}$; x_a is traffic flow on the arc $a \in A$, $x = (\ldots, x_a, \ldots)$; $t_a(x_a)$ is the link travel cost on the arc $a \in A$; $\delta_{a,k}^w$ is the indicator: 1 if acr a is included in route k, 0 if otherwise.

User equilibrium on transportation network G is reached by such x^* that

$$Z(x^*) = \min_x \sum_{a \in A} \int_0^{x_a} t_a(u) du, \qquad (9)$$

subject to

$$\sum_{k \in K^w} f_k^w = F^w, \quad \forall w \in W, \qquad (10)$$

$$f_k^w \geq 0, \quad \forall w \in W, \qquad (11)$$

with definitional constraints

$$x_a = \sum_{w \in W} \sum_{k \in K^w} f_k^w \delta_{a,k}^w, \quad \forall a \in A. \qquad (12)$$

User equilibrium principle allows us to introduce t_w^*, that is equilibrium journey time for any OD-pair w.

Lemma. t_w^* *is the Lagrange multiplier in the optimization program (9)–(12) corresponding to the constraint (11).*

Proof. The Lagrangian of the problem (9)–(12) is

$$L = \sum_{a \in A} \int_0^{x_a} t_a(u)du + \sum_w \mu_w \left(F^w - \sum_{k \in K^w} f_k^w \right) + \\ + \sum_w \sum_{k \in K^w} \eta_k^w \left(-f_k^w \right),$$

where μ_w and $\eta_k^w \geq 0$ are Lagrangian multipliers, and differentiation of the Lagrangian yields:

$$\frac{\partial L}{\partial f_k^w} = \sum_{a \in k} t_a(x_a) - \mu_w - \eta_k^w = 0.$$

Note, that according to complementary slackness $\eta_k^w f_k^w = 0$. Thus, for $f_k^w > 0$ the following expression holds

$$\sum_{a \in k} t_a(x_a) = \mu_w, \quad \forall k \in K^w, w \in W. \qquad (13)$$

Actually, left part of (13) is journey time on any used route ($f_k^w > 0$) between OD-pair r. Therefore, proposition is proved. $\square$

Eventually, according to the Lemma the following equality is true:

$$t_w^* = \sum_{a \in k} t_a(x_a) \quad \forall k \in K^w, w \in W.$$

Theorem 2. *Dual mathemetical problem for a TAP, expressed by (9)–(12), is the following bi-level program:*

$$\max \theta(T) \qquad (14)$$

where $\theta(T)$ is defined by

$$\theta(T) = \min_{f \geq 0} \left\{ \sum_{a \in A} \int_0^{x_a} t_a(s)ds + \\ + \sum_w t_w \left(F^w - \sum_{k \in K^w} f_k^w \right) \right\}, \qquad (15)$$

subject to definitional constraints

$$x_a = \sum_{w \in W} \sum_{k \in K^w} f_k^w \delta_{a,k}^w, \quad \forall a \in A. \qquad (16)$$

Proof. We assume that x^* is a solution of (9)–(12). We introduce multipliers for the flow conservation constraints (10) and (11). Indeed, according to Lemma, we can use t_w as Lagrangian multipliers for the constraints (10). The Lagrangian for the problem (9)–(12) is

$$L(x_a, t_w) = \sum_{a \in A} \int_0^{x_a} t_a(s)ds + \\ + \sum_w t_w \left(F^w - \sum_{k \in K^w} f_k^w \right). \qquad (17)$$

If $L(x_a, t_w)$ has a saddle point (x_a^*, t_w^*) in admissible set, then x_a^* is the solution of the problem (9)–(11), and t_w^* is the solution of the following optimization problem [16]:

$$\max_T L(x_a^*, t_w)$$

in case of

$$x_a = \sum_{w \in W} \sum_{k \in K^w} f_k^w \delta_{a,k}^w, \quad \forall a \in A,$$

where $T = (t_1, \ldots, t_{|W|})^{\mathrm{T}}$.

This duality relation holds when the theorem of equivalence is satisfied [16]. $\square$

4 OD-matrix estimation from plate scanning sensors

Link-flow counts provide the amount of vehicles on the links. Plate scanning sensors associated with the certain links identify plates of vehicles from link-flow. Thus, when any vehicle crosses a link with some sensor then sensor records its plate number and fixation time. Eventually, database consisting of {plate number, fixation time, number of sensor} is accumulated [3]. With the help of such database the travel time between any origin-destination pair can be evaluated directly. Indeed, one just has to know fixation time of the vehicle in origin and fixation time in destination to define t_w^* (as the difference between fixation time in the destination and fixation time in the origin) for any $w \in W$.

Therefore, the following bi-level optimization program can be formulated:

$$\min_F \left(\overline{F} - F \right)^{\mathrm{T}} U^{-1} \left(\overline{F} - F \right) + \\ + (T^* - T)^{\mathrm{T}} (T^* - T), \qquad (18)$$

subject to

$$F \geq 0, \qquad (19)$$

where T solves

$$\max \theta(T), \qquad (20)$$

when $\theta(T)$ is defined by

$$\theta(T) = \min_{f \geq 0} \left\{ \sum_{a \in A} \int_0^{x_a} t_a(s) ds + \right.$$

$$\left. + \sum_w t_w \left(F^w - \sum_{k \in K^w} f_k^w \right) \right\}, \qquad (21)$$

subject to definitional constraints

$$x_a = \sum_{w \in W} \sum_{k \in K^w} f_k^w \delta_{a,k}^w, \quad \forall a \in A. \qquad (22)$$

Here, (18) is the generalized least squares estimation and $\overline{F}$ is the aprior volume of travel demand between all OD-pairs, and U is the weighting matrix.

5 Computational experiment

Let us consider Saint-Petersburg road network (fig. 3). We define seven origin-destination pairs with seven shortest routes from seven periphery origins {1,2,3,4,5,6,7} to the center destination {8}. According to STSI (State Traffic

Figure 3: Selected OD-pairs on the Saint-Petersburg road network with the shortest routes

Safety Inspectorate), nowadays there are 253 plate scanning sensors observing the Saint-Petersburg road network (fig. 4). Due to these sensors, we are able to identify travel time between chosen OD-pairs (table 1). The developed approach is based on user equilibrium principle, which suggests that value of travel time on the shortest route is travel time on any actually used route. Moreover, we are able to calculate aprior flows $\overline{F}$ using the gravity model [4].

Let us use these data as inputs for bi-level optimization program (18)–(22). MATLAB was employed to carry out the simulation. Results of simulation are presented in the table 2. Moreover, these results are available in comparison with aprior flows. Figure 5 gives a visualization of such a comparison. One can see that rough aprior estimation

Figure 4: Sensors location on the Saint-Petersburg road network

Table 1: Journey time obtained from plate scanning sensors

Route in OD-pair	Travel time t^*/min
1–8	89
2–8	80
3–8	83
4–8	78
5–8	45
6–8	57
7–8	36

of trip flows, obtained by gravity model, was adjusted by virtue of information about actual travel time between OD-pairs. Therefore, approach introduced in this paper seems to be quite useful.
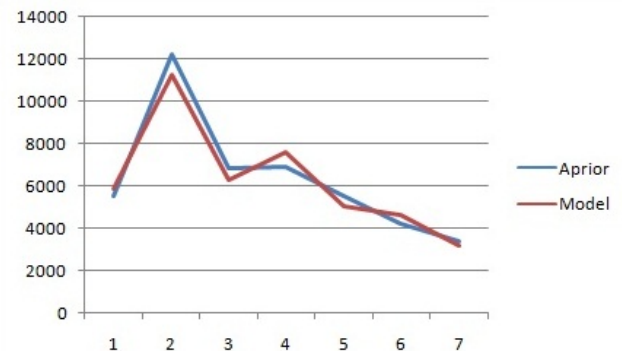

Figure 5: Comparison of model and aprior flows

6 Conclusion

The present paper was devoted to the problem of OD-matrix estimation. Original OD-matrix estimation technique based on a dual formulation of the traffic assign-

Table 2: Comparison of model and aprior flows

OD-pair	Aprior flow	Model flow
1–8	5523	5910
2–8	12232	11253
3–8	6827	6295
4–8	6938	7631
5–8	5534	5080
6–8	4254	4650
7–8	3395	3202

ment problem was offered. Due to this technique traffic demand estimation between certain OD-pair could be solely based on the information about journey time. Since journey time is easily obtained from plate scanning sensors, the developed technique has an obvious practical significance. Moreover, explicit relationship between traffic demand and journey time was obtained for the network of parallel routes with one OD-pair. Such a result gives a clear understanding of basis relationship between demand and journey time. Eventually, the developed approach was experimentally implemented to the Saint-Petersburg road network, that demonstrates its effectiveness.

References

[1] Hazelton M. Inference for origin-destination matrices: estimation, prediction and reconstruction // Transportation Research Part B. 2001. No 35. P. 667–676.

[2] Yang H., Sasaki T., Iida Y., Asakura Y. Estimation of origin-destination matrices from link traffic counts on congested networks // Transportation Research Part B. 1992. No 26 (6). P. 417–434.

[3] Zakharov V., Krylatov A. OD-matrix estimation based on plate scanning // 2014 International Conference on Computer Technologies in Physical and Engineering Applications (ICCTPEA). 2014. P. 209-210.

[4] Medina A., Taft N., Salamatian K., Bhattacharyya S., Diot C. Traffic matrix estimation: existing techniques and new directions // Computer Communication Review – Proceedings of the 2002 SIGCOMM conference. 2002. No 32. P. 161–174.

[5] Castillo E., Menedez J. M., Jimenez P. Trip matrix and path flow reconstruction and estimation based on plate scanning and link observations // Transportation Research Part B. 2008. No 42. P. 455–481.

[6] Bianco L., Cerrone C., Cerulli R., Gentili M. Locating sensors to observe network arc flows: exact and heuristic approaches // Computers and Operation Research. 2014. No 46. P. 12–22.

[7] Minguez R., Sanchez-Cambronero S., Castillo E., Jimenez P. Optimal traffic plate scanning location for OD trip matrix and route estimation in road networks // Transportation Research Part B. 2010. No 44. P. 282–298.

[8] Bierlaire M. The total demand scale: a new measure of quality for static and dynamic origin–destination trip tables // Transportation Research Part B. 2002. No 36. P. 282–298.

[9] Krylatov A. Yu. Optimal strategies for traffic flow management on the transportation network of parallel links // Vestnik of St. Petersburg State University. Series 10. Applied Mathematics. Computer Science. Control Processes. 2014. No 2. P. 121–130.

[10] Patriksson M. The Traffic Assignment Problem: Models and Methods. Utrecht, Netherlands: VSP, 1994. 223 p.

[11] Wardrop J. G. Some theoretical aspects of road traffic research // Proc. Inst. Civil Eng. 1952. Vol. 1, No 3. P. 325–362.

[12] Zakharov V. V., Krylatov A. Yu. Transit Network Design for Green Vehicles Routing // Advances in Intelligent Systems and Computing. 2015. Vol. 360. P. 449–458.

[13] Zakharov V. V., Krylatov A. Yu. Competitive routing of traffic flows by navigation providers // Automation and Remote Control, 2016. Vol. 77, No 1. P. 179–189.

[14] Shavidze G. G., Balykina Y. E., Lejnina E. A., Svirkin M. V. Mathematical Model of Ambulance Resources in Saint-Petersburg // Proceedings of the International Conference on Numerical Analysis and Applied Mathematics. 2015. Vol. 1738.

[15] Krylatov A. Yu. Network flow assignment as a fixed point problem // Journal of Applied and Industrial Mathematics. 2016. No 10 (2). P. 243-256.

[16] Belolipetskiy A.A., Gorelik V. A. Economic and mathematical methods Moscow, Russia: Academiya, 2010. 368 p.