A New Divergence Measure for Intuitionistic Fuzzy Matrices

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Distance / similarity measures facilitate in decision-making by identifying the discrimination or similarity between two or more course of actions to identify the best decision. Data available in the real world may not be in a crisp format. Intuitionistic fuzzy matrices are applicable managing uncertainty and useful in decision making, relational equation, clustering, etc. To take decision the divergence between two intuitionistic fuzzy matrices identifies dissimilarity between matrices. This paper presents a new divergence measure for intuitionistic fuzzy matrices with the verification of its validity. The fundamental properties are demonstrated for the new intuitionistic fuzzy divergence measure. A technique to solve multi-criteria decision-making problems is developed by utilizing the proposed intuitionistic fuzzy divergence measure. Finally, application in the medical diagnosis of this intuitionistic fuzzy divergence measure to decision making is shown using real data.

Povzetek: Razvita je nova mera podobnosti mehkih matrik. Nova metoda omogoča reševanje problemov večkriterijskega odločanja in se uporablja v medicinski diagnostiki z uporabo realnih podatkov.

1 Introduction

In real-world data involved in medical sciences, social sciences, engineering, and management are usually not all crisp, meticulous, and deterministic due to various uncertainties lying in the problem. Ambiguity may not be handled only dealing with fuzziness to reduce uncertainty of fuzzy values. Fuzzy sets use only crisp number for the membership value. However, in some situations the exact value of the function cannot be easily obtained. To tackle this issue, we should consider intuitionistic fuzzy data for linguistic evaluation for more reliable results than fuzzy data which allocates the degrees of membership and non-membership and provide more degree of flexibility to the expert for expressing his judgment. In recent times, due to increasing complexity of socio-economic environment, intuitionistic fuzzy sets are more eminent than the fuzzy sets in maintaining the ambiguity and exaggeration of real-life problems.

In information theory, while studying the set of objects we must differentiate or discriminate two objects on the basis of some parameters associated with them. Divergence measure is a significant tool for evaluating the amount of discrimination. Shannon (1948) defined entropy measure for probability distribution. Kullback and Leibler (1951) firstly investigated divergence measure which assumed probability distribution deviates from the original one found applications in many studies. Bhandari and Pal (1993) give a fuzzy divergence information measure for discrimination of a fuzzy set A relative to some other fuzzy set B which has found wide applications in many fields such as image processing, signal processing, Fuzzy clustering, and pattern recognition etc.

Atanassov (1986) proposed a generalization of fuzzy set, an intuitionistic fuzzy set, is characterized by two functions expressing the degree of membership and non-membership and a hesitation index. IFSs have widely implemented by the researchers to examine problems with uncertainties. Intuitionistic fuzzy measures are more appropriate in decision making such as in medical diagnosis (Wei et al. (2011) and Wu and Zhang (2011)), engineering, speech recognition, pattern recognition (Li and Cheng, (2002); Mitchell (2003); Vlachos and Sergiadis (2007); Wei and Ye (2010)) and in many more areas. Wei and Ye (2010) proposed an improved version of Vlachos and Sergiadis (2007) intuitionistic fuzzy divergence and applied in pattern recognition. Later, various research scholars have paid attention on divergence measure for IFSs (Li and Cheng (2002); Vlachos and Sergiadis (2007); Verma and Maheshwari (2017)). Yue and Jia (2017) investigated a GDM method based on projection measurement for IFSs to solve complex decision-making problem. He et al. (2020) proposed distance measures on IFSs based on IF dissimilarity functions and successfully applied on pattern recognition. Taruna et al. (2021) studied fuzzy distance measure and applied successfully in medical diagnosis. Taruna et al. (2021) proposed a new parametric generalized exponential entropy measure on Intuitionistic vague sets. Arora and Naithani (2021) proposed logarithmic entropy measures under PFSs and demonstrate application proposed measures in detecting disease related to Post Covid-19 implications through TOPSIS method.

Mishra et al. (2018) provides me direction to propose this work, they proposed an intuitionistic fuzzy divergence measure based on ELECTRE method for the performance of cellular mobile telephone service providers. The concept of intuitionistic fuzzy matrix was studied by Pal et al. (2002). Intuitionistic fuzzy matrices (IFM) generalize the fuzzy matrix presented by Thomson (2005) and has been more applicable in multi-criteria decision, bioinformatics, aircraft control and many related fields. It also helps in generalization of various results on fuzzy matrices and can be used in discussion in intuitionistic fuzzy relations. IFM can be used in linear intuitionistic fuzzy transformations whereas it can be helpful in non-linear transformations.

It is noticeable that the strength of a measure lies in its properties. The proposed measure has elegant properties proved in the paper, to enhance the applicability of this measure. Inspired by the above-mentioned work, we propose a divergence measure for intuitionistic fuzzy matrices for applying in multi criteria decision making in every field of real world like in medical, engineering, business where decision have to be taken on the basis of some criteria. The aim of the measure is to evaluate the optimal alternative under the set of the different ones.

The remainder of the paper is organized as follows. Section 2 is devoted to introducing some well-known concepts, and notions related to fuzzy set theory, intuitionistic fuzzy set theory and intuitionistic fuzzy matrix theory. In section 3, we proposed a new intuitionistic fuzzy divergence measure for intuitionistic fuzzy matrix corresponding to Mishra et al. (2018). Section 4 provides more elegant properties of the proposed measure. It is followed by the applications of the proposed intuitionistic fuzzy divergence measure for the IFM to medical diagnosis and a method of multi-criteria decision making in section 5. Finally, some concluding remarks are drawn in section 6.

2 Preliminaries

This section is devoted to introducing some well-known concepts and the notions of fuzzy set theory and intuitionistic fuzzy set theory. Then, we recall the concepts and the notions related to fuzzy set theory and the axiomatic definition of intuitionistic fuzzy divergence measure.

2.1 Definitions and preliminaries

2.1.1 Fuzzy set

The linguistic values of the alternative's assessment are usually symbolized by fuzzy sets for dealing with fuzziness of real –world problem. Fuzzy sets are the sets whose elements have a degree of membership. Zadeh (1965) acquainted the fuzzy sets as the extension of the classical notion of sets i.e. crisp sets.

Definition: A fuzzy set is a pair (Ủ, ḿ), where Ủ is a set and ḿ: Ủ → [0, 1] for each ỳ∈ Ủ, the value ḿ(ỳ) is called degree of membership of ỳ in (Ủ, ḿ). For a finite set Ủ = $\{ỳ_1, ỳ_2, \ldots ỳ_n\}$ the fuzzy set (Ủ, ḿ) is often denoted by

$$\left\{\left\{\frac{ḿ(ỳ_1)}{ỳ_1}\right\}, \left\{\frac{ḿ(ỳ_2)}{ỳ_2}\right\}, \ldots \left\{\frac{ḿ(ỳ_n)}{ỳ_n}\right\}\right\}.$$

Definition: A fuzzy set ḿ(ỳ) on Ủ is defined by a membership function ḿ(ỳ): Ủ → [0,1]. For ỳ∈ Ủ, ḿ(ỳ) the membership function denotes the degree to which ỳ belongs to fuzzy set ḿ.

2.1.2 Intuitionistic fuzzy set

In some situation, fuzzy sets are not efficient for linguistic evaluation. Atanassov (1986) introduced generalization of fuzzy sets which can be applied in such situation. Intuitionistic fuzzy sets (IFSs) allocate degrees of membership and non-membership which give more degrees of flexibility to the decision expert for express her/ his evaluation. They can be implemented to model situations in which fuzzy sets do not provide all the available information.

Definition (Atanassov, 1986) Let $Ü = \{ u_1, u_2, \ldots, u_n\}$ be a discourse set, then an IFS Ġ on Ü is defined as $Ġ = \{\langle μ_i,\ μ_{Ġ}(μ_i),\ v_{Ġ}(μ_i) | μ_i \in Ü\rangle\}$,

Where $μ_{Ġ} :\ Ü \to [0,1]$ and $v_{Ġ} :\ Ü \to [0,1]$ show the membership degree and non-membership degree of $μ_i$ to Ġ in Ü, respectively, with the condition $0 \le μ_{Ġ}(μ_i) \le 1$, $0 \le v_{Ġ}(μ_i) \le 1$ and $0 \le μ_{Ġ}(μ_i) + v_{Ġ}(μ_i) \le 1$, $μ_i ∈ Ü$.

Here, the intuitionistic index (hesitancy degree) of an element $μ_i \in Ü$ to Ġ is given by $π_{Ġ}(μ_i) = 1 - μ_{Ġ}(μ_i) - v_{Ġ}(μ_i)$ and $0 \le π_{Ġ}(μ_i) \le 1$.

2.1.3 Intuitionistic fuzzy matrix theory

The intuitionistic fuzzy matrix theory is very simple and easily applicable in circumstances where agreeness is not sufficient we have to consider disagreeness for the same. The algorithms and algebra for intuitionistic fuzzy matrix theory are applicable for data related problems. Social scientists apply this approach to analyze interactions between attributes and to analyze other analytical tools.

Definition: intuitionistic fuzzy matrix: A intuitionistic fuzzy matrix(IFM)A of order $m \times n$ is defined as $A = \left[< a_{ij\mu},\ a_{ijv} >\right]_{m\times n}$ where $0 \leq a_{ij\mu} \leq 1$is the membership value of the element a_{ij} in A and $0 \leq a_{ijv} \leq 1$ be the non-membership value of element a_{ij} in matrix A maintaining the condition $0 \leq a_{ij\mu} + a_{ijv} \leq 1$. For our convenience, we write A as $A = \left[(a_{ij\mu},\ a_{ijv})\right]_{m\times n}$. We can define an IFM

$$A = \left[(a_{ij\mu},\ a_{ijv})\right]_{m\times n} = \begin{bmatrix} (\mu_{11}, v_{11}) & (\mu_{12}, v_{12}) & \cdots & (\mu_{1n}, v_{1n}) \\ (\mu_{21}, v_{21}) & (\mu_{22}, v_{22}) & \cdots & (\mu_{2n}, v_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}, v_{m1}) & (\mu_{m2}, v_{m2}) & \cdots & (\mu_{mn}, v_{mn}) \end{bmatrix}$$

Example 2.3.1: Suppose a person wants to buy a mobile phone. He selected three companies to purchase a mobile phone, but he wants only one mobile phone of any of the following (Samsung, Mi, Vivo) three companies which suited him best according to features provided by manufacturer. There are three features (cost, camera, storage capacity) on the basis he can select best alternative for him out of those he has chosen. He assigns membership and non-membership values as follows:

$$A = \begin{bmatrix} (0.1,0.5) & (0.3,0.7) & (0.4,0.6) \\ (0.7,0.2 & (0.4,02) & (0.5,0.3) \\ (0.2,0.6) & (0.1,0.7) & (0.9,0.0) \end{bmatrix}$$

is3 $\times$ 3 intuitionistic fuzzy matrix.

2.1.4 Boolean intuitionistic fuzzy matrix

An IFM $B = \left[(b_{ij\mu},\ b_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}$, is a Boolean intuitionistic fuzzy matrix of order $m \times n$ if all the elements of B are either $0\ or\ 1$ maintaining the condition $0 \leq b_{ij\mu} + b_{ijv} \leq 1$. For example,

$$B = \begin{bmatrix} (0,1) & (1,0) & (1,0) \\ (1,0) & (0,1) & (0,1) \end{bmatrix}$$

is2 $\times$ 3 boolean intuitionistic fuzzy matrix.

2.1.5 Most intuitionistic fuzzy matrix

An intuitionistic fuzzy matrix $F = \left[(f_{ij\mu},\ f_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}$, is said to be a most IFM of order $m \times n$ if all the elements of F are equal to 0.5 maintaining the condition $0 \leq f_{ij\mu} + f_{ijv} \leq 1$. For example,

$$F = \begin{bmatrix} (0.5,0.5) & (0.5,0.5) \\ (0.5,0.5) & (0.5,0.5) \\ (0.5,0.5) & (0.5,0.5) \end{bmatrix}$$

is3 $\times$ 2 most intuitionistic fuzzy matrix.

2.1.6 Rectangle intuitionistic fuzzy matrix

Let $R = \left[(r_{ij\mu},\ r_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}$, If $m \neq n$, then R is called an intuitionistic fuzzy rectangular matrix.

2.1.7 Square intuitionistic fuzzy matrix

Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}$, If $m = n$, then S is called an intuitionistic fuzzy square matrix.

2.1.8 Row-Intuitionistic Fuzzy Matrix:

Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}$, If $m = 1$, then S is called an intuitionistic fuzzy row matrix. For example,

$$S = [(0.1,0.8) \quad (0.5,0.3) \quad (0.8,0.1)]$$

is 1 $\times$ 3 row matrix.

2.1.9 Column-intuitionistic fuzzy matrix

Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}$, If $n = 1$, then S is called a intuitionistic fuzzy column matrix. For example,

$$S = \begin{bmatrix} (0.2,0.5) \\ (0.7,0.1) \\ (0.5,0.4) \end{bmatrix}$$

is 3 $\times$ 1 column fuzzy matrix.

2.1.10 Intuitionistic fuzzy diagonal matrix

Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}$,If $n = m$ and $s_{ij\mu} = 0$, $s_{ijv} = 1$ for all $i \neq j$ then S is called a intuitionistic fuzzy diagonal matrix. For example,

$$S = \begin{bmatrix} (0.4,0.2) & (0,1) & (0,1) \\ (0,1) & (0.7,0.1) & (0,1) \\ (0,1) & (0,1) & (0.2,0.5) \end{bmatrix}$$

is 3×3 fuzzy diagonal matrix.

2.1.11 Scalar intuitionistic fuzzy matrix

Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}$,If $n = m$, and $s_{ij\mu} = 0$, $s_{ijv} = 1$for all $i \neq j$ and $s_{ij\mu} = k$, $s_{ijv} = p \in [0, 1] \forall i = j$ then S is called a fuzzy scalar matrix. For example

$$S = \begin{bmatrix} (0.4,0.5) & (0,1) & (0,1) \\ (0,1) & (0.4,0.5) & (0,1) \\ (0,1) & (0,1) & (0.4,0.5) \end{bmatrix}$$

is3×3 fuzzy scalar matrix.

2.1.12 Upper triangular intuitionistic fuzzy matrix

Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}$, where $s_{ij} = (s_{ij\mu},\ s_{ijv})$, If $m = n$, and $s_{ij} = (0,1)$ for all $i > j$ then S is called a intuitionistic fuzzy upper triangular matrix. For example,

$$S = \begin{bmatrix} (0.4,0.1) & (0.5,0.2) & (0.9,0.0) \\ (0,1) & (0.2,0.3) & (0.3,0.4) \\ (0,1) & (0,1) & (0.4,0.5) \end{bmatrix}$$

is 3×3 upper triangular matrix.

2.1.13 Lower triangular intuitionistic fuzzy matrix

Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}$, where $s_{ij} = (s_{ij\mu},\ s_{ijv})$, If $m = n$, and $s_{ij} = (0,1)$for all $i < j$ then S is a intuitionistic fuzzy lower triangular matrix. For example,

$$S = \begin{bmatrix} (0.4,0.3) & (0,1) & (0,1) \\ (0.5,0.4) & (0.9,0.0) & (0,1) \\ (0.3,0.6) & (0,1) & (0.1,0.6) \end{bmatrix}$$

is3×3 fuzzy lower triangular matrix.

An intuitionistic fuzzy matrix is triangular if it is either intuitionistic fuzzy lower or intuitionistic fuzzy upper triangular matrix.

Definition: Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}$,where$s_{ij} = (s_{ij\mu},\ s_{ijv})$,then the elements $s_{11}, s_{22}, \dots, s_{mm}$ are the diagonal elements and which they lie along the line is called the principal diagonal of the intuitionistic fuzzy matrix.

2.2 Operations on two intuitionistic fuzzy matrices

Here we performed some operation on Intuitionistic fuzzy matrices. Let us define two intuitionistic fuzzy matrices S and T of order 3×3 as

$$S = \begin{bmatrix} (0.7,0.0) & (0.5,0.2) & (0.2,0.0) \\ (0.3,0.7) & (0.2,0.3) & (0.1,0.7) \\ (0.3,0.5) & (0.7,0.1) & (0.6,0.2) \end{bmatrix} \quad (2.4.1)$$

$$T = \begin{bmatrix} (0.2,0.5) & (0.3,0.6) & (0.1,0.5) \\ (0.5,0.3) & (0.2,0.6) & (0.6,0.3) \\ (0.8,0.0) & (0.7,0.1) & (1.0,0.0) \end{bmatrix} \quad (2.4.2)$$

2.2.1 Addition and subtraction of two fuzzy matrices

The addition of two intuitionistic fuzzy matrices is possible only when they have same order. Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}, T = \left[t_{ij\mu}, t_{ijv}\right] \in [\text{ÏF(M)}]_{m\times n}$. Then we define addition and subtraction of intuitionistic fuzzy matrices of $S\ and\ T$as:

$$S + T = \left\{\max\left(s_{ij\mu}, t_{ij\mu}\right), \min\left(s_{ijv}, t_{ijv}\right)\right\} \forall i \text{ and } j.$$

$$S - T = \left\{\min\left(s_{ij\mu}, t_{ij\mu}\right), \max\left(s_{ijv}, t_{ijv}\right)\right\} \forall\ i \text{ and } j.$$

The addition of above two matrices given by (2.4.1) and (2.4.2) is possible because they have same order and

$$S + T = \begin{bmatrix} (0.7,0.0) & (0.5,0.2) & (0.2,0.0) \\ (0.5,0.2) & (0.2,0.3) & (0.6,0.3) \\ (0.8,0.0) & (0.7,0.1) & (1.0,0.0) \end{bmatrix}$$

Clearly $S + T$ is alsoan intuitionistic fuzzy matrix. However, addition of two standard IFMs is an intuitionistic fuzzy matrix.

$$S - T = \begin{bmatrix} (0.2,0.5) & (0.3,0.6) & (0.1,0.5) \\ (0.3,0.7) & (0.2,0.6) & (0.1,0.7) \\ (0.3,0.5) & (0.7,0.1) & (0.6,0.2) \end{bmatrix}$$

2.2.2 Product of two fuzzy matrices

We need to define an operation corresponding to multiplication of two intuitionistic fuzzy matrices so that the product again happens to be an intuitionistic fuzzy matrix. The two types of operation which we have are max-min operation and min-max operation. We define multiplication of S and T as

$$S * T = R\ =\ \left[r_{ij\mu},\ r_{ijv})\right]_{m\times n} = \left\{\ maxmin\left((s_{ij\mu}), (t_{ij\mu})\right), minmax\left((s_{ijv}), (t_{ijv})\right)\right\} \forall\ \ i\ and\ j\ .$$

The product of two intuitionistic fuzzy matrices S and T given by (2.4.1) and (2.4.2) is possible because they are square matrices of same order.

Max-min Operation of Two Intuitionistic Fuzzy Matrices (taking membership value)

Definition: Let $S = \left[s_{ij\mu}\right] \in [\text{ÏF(M)}]_{m\times n} \& T = \left[t_{ij\mu}\right] \in [\text{ÏF(M)}]_{n\times m}$ Then Max-min operation of S, T on μ is defined by $Max - min(S_{m\times n}, T_{n\times m}) = R_{m\times m} = [r_{ij\mu}]_{m\times m}$, where $r_{ij\mu} = \max\left\{\min[\left(s_{ij\mu}, t_{ij\mu}\right) forj = 1\ to\ n\ ]\right\} for\ i = 1\ to\ m.$

Example: The Max-min operation of two matrices given by (2.4.1) and (2.4.2) is

$$Max - min(S_{3\times 3}, T_{3\times 3}) = R_{3\times 3} = \begin{bmatrix} 0.3 & 0.5 & 0.7 \\ 0.2 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.7 \end{bmatrix}$$

Where,

$$r_{11} = max\{\min(0.7, 0.2), \min(0.5, 0.3), \min(0.2, 0.1)\} = max\{0.2, 0.3, 0.1\} = 0.3$$

$$r_{12} = max\{\min(0.7, 0.5), \min(0.5, 0.2), \min(0.2, 0.6)\} = max\{0.5, 0.2, 0.2\} = 0.5.$$

and so on

Min-Max Operation of Two Intuitionistic Fuzzy Matrices (non-membership value)

Definition: Let $S = \left[s_{ijv}\right] \in [\text{ÏF(M)}]_{m\times n} \& T = \left[t_{ijv}\right] \in [\text{ÏF(M)}]_{n\times m}$ Then Min-man operation of S, T is defined by $Min - max(S_{m\times n}, T_{m\times n}) = R_{m\times m} = [r_{ijv}]_{m\times m}$, where $r_{ijv} = \min\left\{\max[\left(s_{ijv}, t_{jiv}\right) forj = 1\ ton]\right\} fori = 1\ tom.$

Example: The Min-max operation of two matrices given by (2.4.1) and (2.4.2) is

$$Min - max(S_{3\times3}, T_{3\times3}) = R_{3\times3} = \begin{bmatrix} 0.5 & 0.3 & 0.0 \\ 0.6 & 0.6 & 0.3 \\ 0.5 & 0.3 & 0.1 \end{bmatrix}$$

where,

$$r_{11} = min\{\max(0.0, 0.5), \max(0.2, 0.6), \max(0.0, 0.5)\} = min\{0.5, 0.6, 0.5\} = 0.5$$

$$r_{12} = min\{\max(0.0, 0.3), \max(0.2, 0.6), \max(0.0, 0.3)\} = min\{0.3, 0.6, 0.3\} = 0.3.$$

and so on.

Here we find max-min and min-max products of membership and non-membership value of the matrices respectively to define the product of matrices S and T thus the product matrix R is

$$R = \begin{bmatrix} (0.3,0.5) & (0.5,0.3) & (0.7,0.0) \\ (0.2,0.6) & (0.3,0.6) & (0.3,0.3) \\ (0.3,0.5) & (0.6,0.3) & (0.7,0.1) \end{bmatrix}$$

2.2.3 Conjugate (complement) of intuitionistic fuzzy matrix

Definition: Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}, T = \left[t_{ij\mu}, t_{ijv}\right] \in [\text{ÏF(M)}]_{m\times n}$.then R is conjugate (complement) of S denoted by $S^c = R = \left[r_{ij}\right]$, where $r_{ij} = (\ s_{ijv},\ s_{ij\mu})$ for all i and j.

Example: The conjugate (complement) operation of matrix given by (2.4.1) is

$$S^c = R = S = \begin{bmatrix} (0.0,0.7) & (0.2,0.5) & (0.0,0.2) \\ (0.7,0.3) & (0.3,0.2) & (0.7,0.7) \\ (0.5,0.3) & (0.1,0.7) & (0.2,0.6) \end{bmatrix}$$

2.2.4 Transpose of intuitionistic fuzzy matrix

Definition: Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}, R = \left[r_{ij\mu}, r_{ijv}\right] \in [\text{ÏF(M)}]_{m\times n}$,where $s_{ij} = (s_{ij\mu},\ s_{ijv})$, then R is transpose of S denoted by $S^T = R = \left[r_{ji}\right]$, where $r_{ji} = s_{ij}$ for all i and j.

Example: The transpose operation of matrix given by (2.4.1) is

$$S^T = R = S = \begin{bmatrix} (0.7,0.0) & (0.3,0.7) & (0.3,0.5) \\ (0.5,0.2) & (0.2,0.3) & (0.7,0.1) \\ (0.2,0.0) & (0.1,0.7) & (0.6,0.2) \end{bmatrix}$$

2.2.5 Union of two matrix

Definition: Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}, R = \left[r_{ij\mu}, r_{ijv}\right] \in [\text{ÏF(M)}]_{m\times n}$, where $s_{ij} = (s_{ij\mu},\ s_{ijv})$,

$$S \cup R = \{\left[s_{ij\mu}, s_{ijv}\right] \cup \left[r_{ij\mu}, r_{ijv}\right]\}$$

$$= \{max(s_{ij\mu}, r_{ij\mu}), min(s_{ijv}, r_{ijv})\}$$

2.2.6 Intersection of two matrix

Definition: Let $S = \left[(s_{ij\mu},\ s_{ijv})\right]_{m\times n} \in [\text{ÏF(M)}]_{m\times n}, R = \left[r_{ij\mu}, r_{ijv}\right] \in [\text{ÏF(M)}]_{m\times n}$,where $s_{ij} = (s_{ij\mu},\ s_{ijv})$,

$$S \cap R = \left[s_{ij\mu}, s_{ijv}\right] \cap \left[r_{ij\mu}, r_{ijv}\right]$$

$$= \left[min(s_{ij\mu}, r_{ij\mu}), max(\ s_{ijv}, r_{ijv})\right]$$

3 Intuitionistic fuzzy divergence measueres

Firstly, De Luca and Termini (1972) proposed entropy function for fuzzy set A is defined as follows:

$$H_{DL} = \frac{-1}{n}\sum_{i=1}^{n}[\mu_A(x_i) ln\, \mu_A(x_i) + (1 - \mu_A(x_i)) ln(1 - \mu_A(x_i))]$$

Later, in 1989 Pal and Pal pioneered exponential entropy for fuzzy set A, which is as follows:

$$H_{PP}(A) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}[\, \mu_A(x_i) e^{(1-\mu_A(x_i))} + (1 - \mu_A(x_i)) e^{\mu_A(x_i)} - 1$$

After towards, for the first time to measure the distinction for fuzzy sets Bhandari and Pal (1993) developed divergence measure for fuzzy set.

Let A, B be two fuzzy set, then divergence measure is

$$J_{BP}(A,B) = \sum \mu_A(x_i) ln \frac{\mu_A(x_i)}{\mu_B(y_i)} + (1 - \mu_A(x_i)) ln \frac{1-\mu_A(x_i)}{1-\mu_B(y_i)}$$

Later, Fan and Xie (1999) developed exponential divergence measure for fuzzy set is as follows:

$$J_{FX}(A,B) = \sum_{i=1}^{n}[\{1 - (1 - \mu_A(x_i)) e^{(\mu_A(x_i) - \mu_B(y_i)} + (1 - \mu_A(x_i)) e^{(\mu_B(y_i) - \mu_A(x_i))}\}]$$

Vlachos and Sergiadis (2007) introduced

intuitionistic fuzzy divergence measure is as follows:

$$D_{VS}(A,B) = \sum_{i=1}^{n}\left[\mu_A(x_i) ln\left(\frac{\mu_A(x_i)}{\left(\left(\frac{1}{2}\right)(\mu_A(x_i) + \mu_B(y_i))\right)}\right) + \nu_A(x_i) ln\left(\frac{\mu_A(x_i)}{\left(\left(\frac{1}{2}\right)(\nu_A(x_i) + \nu_B(y_i))\right)}\right)\right]$$

Verma and Maheshwari (2017) defined the Jenson Shannon divergence measure for fuzzy sets:

$$H_{VM}(A,B) = \frac{1}{n(\sqrt{e}-1)}\sum_{i=1}^{n}[\frac{\mu_A(x_i) + \mu_B(y_i)}{2} e^{\frac{2-\mu_A(x_i)-\mu_B(y_i)}{2}} + \left(\frac{2 - \mu_A(x_i) - \mu_B(y_i)}{2}\right) e^{\frac{(\mu_A(x_i)+\mu_B(y_i))}{2}} - \frac{1}{2}(\mu_A(x_i) e^{(1-\mu_A(x_i)} + (1 - \mu_A(x_i) e^{\mu_A(x_i)} + \mu_B(y_i) e^{(1-\mu_B(y_i)} + (1 - \mu_B(y_i) e^{\mu_B(y_i)})$$

Symmetric measure of Verma and Maheshwari (2017) is based on given by Hung and Yang (2008) for IFSs is as follows:

$$D_{HY}(A,B) = \frac{1}{1-\rho}\sum_{i=1}^{n}\left[\left(\frac{\mu_A(x_i) + \mu_B(y_i)}{2}\right)^{\rho} - \left(\frac{\mu_A{}^{\rho}(x_i) + \mu_B{}^{\rho}(y_i)}{2}\right) + \left(\frac{\nu_A(x_i) + \nu_B(y_i)}{2}\right)^{\rho} - \left(\frac{\nu_A{}^{\rho}(x_i) + \nu_B{}^{\rho}(y_i)}{2}\right) + \left(\frac{\pi_A(x_i) + \pi_B(y_i)}{2}\right)^{\rho} - \left(\frac{\pi_A{}^{\rho}(x_i) + \pi_B{}^{\rho}(y_{i)}}{2}\right)\right]$$

Where$\rho > 0$

Zhang and Jiang (2008) introduced intuitionistic fuzzy divergence measure is as follows:

$$D_{ZJ}(A,B) = \sum_{i=1}^{n}\left[\left(\frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2}\right) ln \frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{\left(\left(\frac{1}{2}\right)(\mu_A(x_i) - \nu_A(x_i)) + 2 + (\mu_B(x_i) - \nu_B(y_i))\right)} + \left(\frac{\nu_A(x_i) + 1 - \mu_A(x_i)}{2}\right) ln \frac{\nu_A(x_i) + 1 - \mu_A(x_i)}{\left(\left(\frac{1}{2}\right)(\nu_A(x_i) - \mu_A(x_i)) + 2 + (\nu_B(y_i) - \mu_B(y_i))\right)}\right]$$

Mao et al. (2013) introduced an intuitionistic fuzzy divergence measure:

$$D_{JM}(A,B) = \sum_{i=1}^{n}\left[\pi_A(x_i) ln \frac{\pi_A(x_i)}{\left(\left(\frac{1}{2}\right)(\pi_A(x_i) + \pi_B(y_i)\right)} + \Delta_A(x_i) ln \frac{\Delta_A(x_i)}{\left(\left(\frac{1}{2}\right)(\Delta_A(x_i) + \Delta_B(y_i)\right)}\right]$$

Where $\Delta_A(x_i) = |\mu_A(x_i) - \nu_A(x_i)|$

Properties of divergence measure for intuitionistic fuzzy sets

Definition: Let $A \,\&\, B \in IFSs$, then a mapping $\text{Ɗ} : IFSs \times IFSs \rightarrow R$ is said to divergence measure if it holds the postulates:

(D1). $\text{Ɗ}(A, B) = \text{Ɗ}(B, A)$.

(D2). $\text{Ɗ}(A, B) = 0 \; iff \; A = B$.

(D3). $\text{Ɗ}(A \cap C, B \cap C) \leq \text{Ɗ}(A, B) \; for \; every \; C \in IFS(U)$.

(D4). $\text{Ɗ}(A \cup C, B \cup C) \leq \text{Ɗ}(A, B) \; for \; every \; C \in IFS(U)$.

4 Divergence measure for intuitionistic fuzzy matrix

In this section, a divergence measure for intuitionistic fuzzy matrices $A \,\&\, B$ is proposed which is based on Vlachos and Sergiadis (2007). The validity of the proposed measure is also verified.

Here first we define a non-probabilistic divergence measure of intuitionistic fuzzy matrices:

Definition: Let $[\text{ÏƑ}(M)]_{m\times n}$ be the set of all intuitionistic fuzzy matrices having m rows and n columns and $X \,\&\, Y \in [\text{ÏƑ}(M)]_{m\times n}$. Then a mapping $J : [\text{ÏƑ}(M)]_{m\times n} \times [\text{ÏƑ}(M)]_{m\times n} \rightarrow Z$ is called non-probabilistic divergence measure of intuitionistic fuzzy matrices if and only if it follows properties given below:

4.1 Properties of proposed divergence measure for IFMs

Let $A, B, C \in [\text{ÏƑ}(M)]_{m\times n}$, divergence measure $\text{Ɗ}(A, B)$ given by (3.) satisfies the following postulate:

(P1). $\text{Ɗ}(A, B) = \text{Ɗ}(B, A)$.

(P2). $\text{Ɗ}(A, A^c) = 1 \; iff \; A \in [\text{ÏƑ}(M)]_{m\times n}$.

(P3). $\text{Ɗ}(A, B) = 0 \; iff \; A = B \;\; 0 \leq \text{Ɗ}(A, B) \leq 1$.

(P4). $\text{Ɗ}(A, B) = \text{Ɗ}(A^c, B^c)$ and $\text{Ɗ}(A^c, B) = \text{Ɗ}(A, B^c)$.

(P5). $\text{Ɗ}(A, B) \leq \text{Ɗ}(A, C)$ and $\text{Ɗ}(B, C) \leq \text{Ɗ}(A, C)$ for $A \subseteq B \subseteq C$.

(P6). $\text{Ɗ}(A \cup B, A \cap B) = \text{Ɗ}(A, B)$.

(P7). $\text{Ɗ}(A \cup B, C) \leq \text{Ɗ}(A, C) + \text{Ɗ}(B, C) \;\; for \; all \; C \in [\text{ÏƑ}(M)]_{m\times n}$.

(P8). $\text{Ɗ}(A \cap B, C) \leq \text{Ɗ}(A, C) + \text{Ɗ}(B, C) \;\; for \; all \; C \in [\text{ÏƑ}(M)]_{m\times n}$.

(P9). $\text{Ɗ}(A \cap C, B \cap C) \leq \text{Ɗ}(A, B) \; for \; all \; C \in [\text{ÏƑ}(M)]_{m\times n}$.

(P10). $\text{Ɗ}(A \cup C, B \cup C) \leq \text{Ɗ}(A, B) \; for \; all \; C \in [\text{ÏƑ}(M)]_{m\times n}$.

Here we proposed a divergence measure for intuitionistic fuzzy matrices X and Y of order $m \times n$ which is logarithmic in nature as follows:

$$\text{Ɗ}(A, B) = \frac{-1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c} \left(\frac{\mu_A(x_{ij})+\mu_B(y_{ij})}{2}\right) log \frac{\mu_A(x_{ij})+\mu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)} \\ + \left(\frac{\nu_A(x_{ij})+\nu_B(y_{ij})}{2}\right) log \frac{\nu_A(x_{ij})+\nu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)} \\ - \left(\frac{\pi_A(x_{ij})+\pi_B(y_{ij})}{2}\right) \\ -\frac{1}{2}\left\{\begin{array}{c} \mu_A(x_{ij}) log \left(\frac{\mu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right) \\ +\nu_A(x_{ij}) log \left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right) \\ +\mu_B(y_{ij}) log \left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right) \\ +\nu_B(y_{ij}) log \left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right) - \left(\pi_A(x_{ij}) + \pi_B(y_{ij})\right) \end{array}\right\} \end{array}\right] \quad (4.1)$$

where $\mu_A(x_{ij}), \nu_A(x_{ij}) \in X \;\&\; \mu_B(y_{ij}), \nu_B(y_{ij}) \in Y$

Now to show that our proposed measure is a valid measure since it satisfies all the above axioms which are proved in the following theorems:

Theorem (P1): $\text{Ɗ}(A\!:\!B)$ is symmetric measure i.e. $\text{Ɗ}(A\!:\!B) = \text{Ɗ}(B\!:\!A)$.

Proof: To prove $\text{Ɗ}(A\!:\!B)$ is symmetric measure we show $\text{Ɗ}(A\!:\!B) - \text{Ɗ}(B\!:\!A) = 0$

$\mathcal{D}(B,A)$

$$= \frac{-1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left(\frac{\mu_B(y_{ij})+\mu_A(x_{ij})}{2}\right)log\frac{\mu_B(y_{ij})+\mu_A(x_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_A(x_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)}\right.$$

$$+\left(\frac{\nu_B(y_{ij})+\nu_A(x_{ij})}{2}\right)log\frac{\nu_B(y_{ij})+\nu_A(x_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_A(x_{ij})\right)+\nu_B(y_{ij})+\nu_A(x_{ij})\right)}$$

$$\left.-\left(\frac{\pi_B(y_{ij})+\pi_A(x_{ij})}{2}\right)-\frac{1}{2}\left\{\begin{array}{c}+\mu_B(y_{ij})log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)\\+\nu_B(y_{ij})log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)\\+\mu_A(x_{ij})log\left(\frac{\mu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\\+\nu_A(x_{ij})log\left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\\-\left(\pi_B(y_{ij})+\pi_A(x_{ij})\right)\end{array}\right\}\right]$$

$\mathcal{D}(A,B)-\mathcal{D}(B,A)$

$$= \frac{-1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\left(\frac{\mu_A(x_{ij})+\mu_B(y_{ij})}{2}\right)log\frac{\mu_A(x_{ij})+\mu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)}\right.$$

$$+\left(\frac{\nu_A(x_{ij})+\nu_B(y_{ij})}{2}\right)log\frac{\nu_A(x_{ij})+\nu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)}$$

$$-\left(\frac{\pi_A(x_{ij})+\pi_B(y_{ij})}{2}\right)$$

$$-\frac{1}{2}\left\{\mu_A(x_{ij})log\left(\frac{\mu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)+\nu_A(x_{ij})log\left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\right.$$

$$+\mu_B(y_{ij})log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)+\nu_B(y_{ij})log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)$$

$$\left.-\left(\pi_A(x_{ij})+\pi_B(y_{ij})\right)\right\}$$

$$-\left(\frac{\mu_B(y_{ij})+\mu_A(x_{ij})}{2}\right)log\frac{\mu_B(y_{ij})+\mu_A(x_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_A(x_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)}$$

$$+\left(\frac{\nu_B(y_{ij})+\nu_A(x_{ij})}{2}\right)log\frac{\nu_B(y_{ij})+\nu_A(x_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_A(x_{ij})\right)+\nu_B(y_{ij})+\nu_A(x_{ij})\right)}$$

$$-\left(\frac{\pi_B(y_{ij})+\pi_A(x_{ij})}{2}\right)$$

$$-\frac{1}{2}\left\{\mu_B(y_{ij})log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)+\nu_B(y_{ij})log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)\right.$$

$$+\mu_A(x_{ij})log\left(\frac{\mu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)+\nu_A(x_{ij})log\left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)$$

$$\left.\left.-\left(\pi_B(y_{ij})+\pi_A(x_{ij})\right)\right\}\right]$$

$$=\frac{-1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}[0].$$

Hence, we can say that the divergence measure is symmetric in nature.

Theorem (P2): $\mathcal{D}(\mathrm{A},\mathrm{B})=0\ iff\ \ A=B \qquad 0\leq \mathcal{D}(\mathrm{A},\mathrm{B})\leq 1.$

Theorem (P3): $\mathcal{D}(\mathrm{A},\mathrm{A}^{c})=1\ iff\ \mathrm{A}\in[\text{ÏF}(\mathrm{M})]_{m\times n}$.

Proof: **S**ince $\mathcal{F}(x)=xlogx\ and\ 0\leq x\leq 1$,then $\mathcal{F}'(x)=1+logx$ and $\mathcal{F}'(x)=\frac{1}{x}\succ 0.$

As a result, $\mathcal{F}$is a convex function of X. Therefore, for any two points $x\ and\ y$ in (0,1] the following inequality holds:$\frac{\mathcal{F}(x)+\mathcal{F}(y)}{2}\leq\mathcal{F}\left(\frac{x+y}{2}\right)$. Consequently, $xlogx+ylogy-(x+y)\log\left(\frac{x+y}{2}\right)\geq 0$, with the equality hold only for $x=y$. In addition, if $x=y=0$ the equality holds also. Therefore, we observe that the inequality has the same form of [2018]. Hence, it follows that $0\leq\mathcal{D}(A,B)\leq 1,\mathcal{D}(A,B)=0$ if and only if $A=B$ and $\mathcal{D}(\mathrm{A},\mathrm{A}^{c})=1$.

Theorem(P4): $\mathcal{D}(\mathrm{A},\mathrm{B})=\mathcal{D}(\mathrm{A}^{c},\mathrm{B}^{c})$and $\mathcal{D}(\mathrm{A}^{c},\mathrm{B})=\mathcal{D}(\mathrm{A},\mathrm{B}^{c})$.

Proof: (i) $\mathcal{D}(\mathrm{A},\mathrm{B})=\mathcal{D}(\mathrm{A}^{c},\mathrm{B}^{c})$, to prove this first find $\mathcal{D}(\mathrm{A}^{c},\mathrm{B}^{c})$.

$\mathcal{D}(A,B^{c})$

$$=\frac{-1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c}\left(\frac{\mu_A(x_{ij})+\nu_B(y_{ij})}{2}\right)log\frac{\mu_A(x_{ij})+\nu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\nu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\mu_B(y_{ij})\right)\right)}\\+\left(\frac{\nu_A(x_{ij})+\mu_B(y_{ij})}{2}\right)log\frac{\nu_A(x_{ij})+\mu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\nu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\mu_B(y_{ij})\right)\right)}\\-\left(\frac{\pi_A(x_{ij})+\pi_B(y_{ij})}{2}\right)\\-\frac{1}{2}\left\{\begin{array}{c}\mu_A(x_{ij})log\left(\frac{\mu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\\+\nu_A(x_{ij})log\left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\\+\nu_B(y_{ij})log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)\\+\mu_B(y_{ij})log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)\\-\left(\pi_A(x_{ij})+\pi_B(y_{ij})\right)\end{array}\right\}\end{array}\right]$$

$$= \frac{-1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Bigg[\left(\frac{\nu_A(x_{ij})+\nu_B(y_{ij})}{2}\right)log\frac{\nu_A(x_{ij})+\nu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)}$$
$$+\left(\frac{\mu_B(y_{ij})+\mu_A(x_{ij})}{2}\right)log\frac{\mu_B(y_{ij})+\mu_A(x_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_A(x_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)}$$
$$-\left(\frac{\pi_B(y_{ij})+\pi_A(x_{ij})}{2}\right)$$
$$-\frac{1}{2}\Bigg\{\mu_A(x_{ij})log\left(\frac{\mu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)+\nu_A(x_{ij})log\left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)$$
$$+\mu_B(y_{ij})log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)+\nu_B(y_{ij})log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)$$
$$-\left(\pi_B(y_{ij})+\pi_A(x_{ij})\right)\Bigg\}\Bigg]$$

$$= \frac{-1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c}\left(\frac{\mu_A(x_{ij})+\mu_B(y_{ij})}{2}\right)log\frac{\mu_A(x_{ij})+\mu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)}\\ +\left(\frac{\nu_A(x_{ij})+\nu_B(y_{ij})}{2}\right)log\frac{\nu_A(x_{ij})+\nu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)}\\ -\left(\frac{\pi_A(x_{ij})+\pi_B(y_{ij})}{2}\right)\\ -\frac{1}{2}\left\{\begin{array}{c}\mu_A(x_{ij})log\left(\frac{\mu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\\ +\nu_A(x_{ij})log\left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\\ +\mu_B(y_{ij})log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)\\ +\nu_B(y_{ij})log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)\\ -\left(\pi_A(x_{ij})+\pi_B(y_{ij})\right)\end{array}\right\}\end{array}\right]$$
$$= \mathcal{D}(A,B)$$

(ii)$\mathcal{D}(A^c, B) = \mathcal{D}(A, B^c)$

Proof: To prove this, taking L.H.S.

$\mathcal{D}(A^c, B)$

$$= \frac{-1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Bigg[\left(\frac{\nu_A(x_{ij})+\mu_B(y_{ij})}{2}\right)log\frac{\nu_A(x_{ij})+\mu_B(y_{ij})}{\left(\left(\nu_A(x_{ij})+\mu_B(y_{ij})\right)+\mu_A(x_{ij})+\nu_B(y_{ij})\right)}$$
$$+\left(\frac{\mu_A(x_{ij})+\nu_B(y_{ij})}{2}\right)log\frac{\mu_A(x_{ij})+\nu_B(y_{ij})}{\left(\left(\nu_A(x_{ij})+\mu_B(y_{ij})\right)+\mu_A(x_{ij})+\nu_B(y_{ij})\right)}$$
$$-\left(\frac{\pi_A(x_{ij})+\pi_B(y_{ij})}{2}\right)$$
$$-\frac{1}{2}\Bigg\{\nu_A(x_{ij})log\left(\frac{\nu_A(x_{ij})}{\nu_A(x_{ij})+\nu_A(x_{ij})}\right)+\mu_A(x_{ij})log\left(\frac{\mu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)$$
$$+\mu_B(y_{ij})log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)+\nu_B(y_{ij})log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)$$
$$-\left(\pi_A(x_{ij})+\pi_B(y_{ij})\right)\Bigg\}\Bigg]$$

Now, taking R.H.S,

Here, from L.H.S and R.H.S it is clear that $\mathcal{D}(A^c, B) = \mathcal{D}(A, B^c)$.

Theorem(P5):$\mathcal{D}(A, B) \leq \mathcal{D}(A, C)$ and $\mathcal{D}(B, C) \leq \mathcal{D}(A, C)$ for $A \subseteq B \subseteq C$.

Proof: Let $A \subseteq B \subseteq C$, then $\mu_A \leq \mu_B \leq \mu_C$ and $\nu_A \geq \nu_B \geq \nu_C$, then

Let $|\mu_A - \mu_B| + |\nu_A - \nu_B| + |\pi_A - \pi_B| \leq |\mu_A - \mu_C| + |\nu_A - \nu_C| + |\pi_A - \pi_C|$,

$|\mu_B - \mu_C| + |\nu_B - \nu_C| + |\pi_B - \pi_C| \leq |\mu_A - \mu_C| + |\nu_A - \nu_C| + |\pi_A - \pi_C|$,

Therefore,

$\mathcal{D}(A, B) \leq \mathcal{D}(A, C)$and $\mathcal{D}(B, C) \leq \mathcal{D}(A, C)$.

For the proof of further properties, we consider IFM G into two parts $G_1$and G_2, such that

$$\left.\begin{array}{l}G_1 = \{x_{ij}\ or\ y_{ij}\ ;\ x_{ij} \in X\ or\ y_{ij} \in Y;\ x_{ij} \geq y_{ij}\}\\ G_2 = \{x_{ij}\ or\ y_{ij}\ ;\ x_{ij} \in X\ or\ y_{ij} \in Y;\ x_{ij} < y_{ij}\}\end{array}\right\} \quad (2)$$

And note that for all $x_{ij}, y_{ij} \in G_1$,

$\mu_A(x_{ij}) \geq \mu_B(y_{ij})and\nu_A(x_{ij}) < \nu_B(y_{ij})$

And also, for all $x_{ij}, y_{ij} \in G_2$,

$\mu_A(x_{ij}) < \mu_B(y_{ij})$and $\nu_A(x_{ij}) \geq \nu_B(y_{ij})$

Theorem(P6):$\mathcal{D}(A \cup B, A \cap B) = \mathcal{D}(A, B)$.

Proof: To prove the result, we should prove the following relation

$$\mathcal{D}(A \cup B, A \cap B) = \mathcal{D}(A \cup B \mid A \cap B) + \mathcal{D}(A \cap B | A \cup B)$$

By using the definition of (1), first we have

$$\mathcal{D}(A\cup B \mid A\cap B) =$$

$$\frac{-1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c}\left(\frac{\mu_{A\cup B}(x_{ij})+\mu_{A\cap B}(y_{ij})}{2}\right)\log\frac{\mu_{A\cup B}(x_{ij})+\mu_{A\cap B}(y_{ij})}{\left(\left(\mu_{A\cup B}(x_{ij})+\mu_{A\cap B}(y_{ij})\right)+\nu_{A\cup B}(x_{ij})+\nu_{A\cap B}(y_{ij})\right)}\\ +\left(\frac{\nu_{A\cup B}(x_{ij})+\nu_{A\cap B}(y_{ij})}{2}\right)\log\frac{\nu_{A\cup B}(x_{ij})+\nu_{A\cap B}(y_{ij})}{\left(\left(\mu_{A\cup B}(x_{ij})+\mu_{A\cap B}(y_{ij})\right)+\left(\nu_{A\cup B}(x_{ij})+\nu_{A\cap B}(y_{ij})\right)\right)}\\ -\left(\frac{\pi_{A\cup B}(x_{ij})+\pi_{A\cap B}(y_{ij})}{2}\right)\\ -\frac{1}{2}\left\{\begin{array}{c}\mu_{A\cup B}(x_{ij})\log\left(\frac{\mu_{A\cup B}(x_{ij})}{\mu_{A\cup B}(x_{ij})+\nu_{A\cup B}(x_{ij})}\right)\\ +\nu_{A\cup B}(x_{ij})\log\left(\frac{\nu_{A\cup B}(x_{ij})}{\mu_{A\cup B}(x_{ij})+\nu_{A\cup B}(x_{ij})}\right)\\ +\mu_{A\cap B}(y_{ij})\log\left(\frac{\mu_{A\cap B}(y_{ij})}{\mu_{A\cap B}(y_{ij})+\nu_{A\cap B}(y_{ij})}\right)\\ +\nu_{A\cap B}(y_{ij})\log\left(\frac{\nu_{A\cap B}(y_{ij})}{\mu_{A\cap B}(y_{ij})+\nu_{A\cap B}(y_{ij})}\right)\\ -\left(\pi_{A\cup B}(x_{ij})+\pi_{A\cap B}(y_{ij})\right)\end{array}\right\}\end{array}\right]$$

$$=\frac{-1}{mn}\sum_{x_{ij},y_{ij}\in G_1}\left[\begin{array}{c}\left(\frac{\mu_A(x_{ij})+\mu_B(y_{ij})}{2}\right)\log\frac{\mu_A(x_{ij})+\mu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)}\\ +\left(\frac{\nu_A(x_{ij})+\nu_B(y_{ij})}{2}\right)\log\frac{\nu_A(x_{ij})+\nu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)}\\ -\left(\frac{\pi_A(x_{ij})+\pi_B(y_{ij})}{2}\right)\\ -\frac{1}{2}\left\{\begin{array}{c}\mu_A(x_{ij})\log\left(\frac{\mu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\\ +\nu_A(x_{ij})\log\left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\\ +\mu_B(y_{ij})\log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)\\ +\nu_B(y_{ij})\log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)-\left(\pi_A(x_{ij})+\pi_B(y_{ij})\right)\end{array}\right\}\end{array}\right]$$

$$+\sum_{x_{ij},y_{ij}\in G_2}\left[\left(\frac{\mu_B(y_{ij})+\mu_A(x_{ij})}{2}\right)\log\frac{\mu_B(y_{ij})+\mu_A(x_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_A(x_{ij})\right)+\left(\nu_B(y_{ij})+\nu_A(x_{ij})\right)\right)}\right.$$
$$+\left(\frac{\nu_B(y_{ij})+\nu_A(x_{ij})}{2}\right)\log\frac{\nu_B(y_{ij})+\nu_A(x_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_A(x_{ij})\right)+\left(\nu_B(y_{ij})+\nu_A(x_{ij})\right)\right)}$$
$$-\left(\frac{\pi_B(y_{ij})+\pi_A(x_{ij})}{2}\right)$$
$$-\frac{1}{2}\left\{\mu_B(y_{ij})\log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)+\nu_B(y_{ij})\log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)\right.$$
$$+\mu_A(x_{ij})\log\left(\frac{\mu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)+\nu_A(x_{ij})\log\left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)$$
$$\left.\left.-\left(\pi_B(y_{ij})+\pi_A(x_{ij})\right)\right\}\right]$$

Now, again from the definition (1), we have

$$D(A\cap B|A\cup B) =$$

$$\frac{-1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c}\left(\frac{\mu_{A\cap B}(x_{ij})+\mu_{A\cup B}(y_{ij})}{2}\right)\log\frac{\mu_{A\cap B}(x_{ij})+\mu_{A\cup B}(y_{ij})}{\left(\left(\mu_{A\cap B}(x_{ij})+\mu_{A\cup B}(y_{ij})\right)+\left(\nu_{A\cap B}(x_{ij})+\nu_{A\cup B}(y_{ij})\right)\right)}\\ +\left(\frac{\nu_{A\cap B}(x_{ij})+\nu_{A\cup B}(y_{ij})}{2}\right)\log\frac{\nu_{A\cap B}(x_{ij})+\nu_{A\cup B}(y_{ij})}{\left(\left(\mu_{A\cap B}(x_{ij})+\mu_{A\cup B}(y_{ij})\right)+\left(\nu_{A\cap B}(x_{ij})+\nu_{A\cup B}(y_{ij})\right)\right)}\\ -\left(\frac{\pi_{A\cap B}(x_{ij})+\pi_{A\cup B}(y_{ij})}{2}\right)\\ -\frac{1}{2}\left\{\begin{array}{c}\mu_{A\cap B}(x_{ij})\log\left(\frac{\mu_{A\cap B}(x_{ij})}{\mu_{A\cap B}(x_{ij})+\nu_{A\cap B}(x_{ij})}\right)\\ +\nu_{A\cap B}(x_{ij})\log\left(\frac{\nu_{A\cap B}(x_{ij})}{\mu_{A\cap B}(x_{ij})+\nu_{A\cap B}(x_{ij})}\right)\\ +\mu_{A\cup B}(y_{ij})\log\left(\frac{\mu_{A\cup B}(y_{ij})}{\mu_{A\cup B}(y_{ij})+\nu_{A\cup B}(y_{ij})}\right)\\ +\nu_{A\cup B}(y_{ij})\log\left(\frac{\nu_{A\cup B}(y_{ij})}{\mu_{A\cup B}(y_{ij})+\nu_{A\cup B}(y_{ij})}\right)\\ -\left(\pi_{A\cap B}(x_{ij})+\pi_{A\cup B}(y_{ij})\right)\end{array}\right\}\end{array}\right]$$

$$\frac{-1}{mn}\sum\left[\sum_{x_{ij},y_{ij}\in G_2}\left[\left(\frac{\mu_B(y_{ij})+\mu_A(x_{ij})}{2}\right)\log\frac{\mu_B(y_{ij})+\mu_A(x_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_A(x_{ij})\right)+\left(\nu_B(y_{ij})+\nu_A(x_{ij})\right)\right)}\right.\right.$$
$$+\left(\frac{\nu_B(y_{ij})+\nu_A(x_{ij})}{2}\right)\log\frac{\nu_B(y_{ij})+\nu_A(x_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_A(x_{ij})\right)+\left(\nu_B(y_{ij})+\nu_A(x_{ij})\right)\right)}$$
$$\left.\left.-\left(\frac{\pi_B(y_{ij})+\pi_A(x_{ij})}{2}\right)-\frac{1}{2}\left\{\begin{array}{c}\mu_B(y_{ij})\log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)\\ +\nu_B(y_{ij})\log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)_A\\ +\mu(x_{ij})\log\left(\frac{\mu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\\ +\nu_A(x_{ij})\log\left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\\ -\left(\pi_B(y_{ij})+\pi_A(x_{ij})\right)\end{array}\right\}\right]\right]$$

$$+\sum\sum_{x_{ij},y_{ij}\in G_2}\left[\begin{array}{c}\left(\frac{\mu_A(x_{ij})+\mu_B(y_{ij})}{2}\right)\log\frac{\mu_A(x_{ij})+\mu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)}\\ +\left(\frac{\nu_A(x_{ij})+\nu_B(y_{ij})}{2}\right)\log\frac{\nu_A(x_{ij})+\nu_B(y_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_B(y_{ij})\right)+\left(\nu_A(x_{ij})+\nu_B(y_{ij})\right)\right)}\\ -\left(\frac{\pi_A(x_{ij})+\pi_B(y_{ij})}{2}\right)-\frac{1}{2}\left\{\begin{array}{c}\mu_A(x_{ij})\log\left(\frac{\mu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\\ +\nu_A(x_{ij})\log\left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right)\\ +\mu_B(y_{ij})\log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)\\ +\nu_B(y_{ij})\log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right)\\ -\left(\pi_A(x_{ij})+\pi_B(y_{ij})\right)\end{array}\right\}\end{array}\right]$$

Finally, by adding these two equations, we get

$$\mathcal{D}(A\cup B, A\cap B) = \mathcal{D}(A\cup B, A\cap B)$$
$$= \mathcal{D}(A\cup B \mid A\cap B) + \mathcal{D}(A\cap B|A\cup B)$$
$$\mathcal{D}(A, B).$$

Theorem (P7): $Ɗ(A\cup B, C) \leq Ɗ(A, C) + Ɗ(B, C)$.

Proof: Consider the expression

$$Ɗ(A, C) + Ɗ(B, C) - Ɗ(A\cup B, C)$$

$$= \frac{-1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c} \left(\frac{\mu_A(x_{ij})+\mu_C(z_{ij})}{2}\right) log \frac{\mu_A(x_{ij})+\mu_C(z_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_C(z_{ij})\right)+\left(\nu_A(x_{ij})+\nu_C(z_{ij})\right)\right)} \\ +\left(\frac{\nu_A(x_{ij})+\nu_C(z_{ij})}{2}\right) log \frac{\nu_A(x_{ij})+\nu_C(z_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_C(z_{ij})\right)+\left(\nu_A(x_{ij})+\nu_C(z_{ij})\right)\right)} \\ -\left(\frac{\pi_A(x_{ij})+\pi_C(z_{ij})}{2}\right) - \frac{1}{2}\left\{\begin{array}{c} \mu_A(x_{ij}) log\left(\frac{\mu_A(x_{ij})}{\mu_{A\cup B}(x_{ij})+\nu_{A\cup B}(x_{ij})}\right) \\ +\nu_A(x_{ij}) log\left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right) \\ +\mu_C(z_{ij}) log\left(\frac{\mu_C(z_{ij})}{\mu_C(z_{ij})+\nu_C(z_{ij})}\right) \\ +\nu_C(y_{ij}) log\left(\frac{\nu_C(z_{ij})}{\mu_C(z_{ij})+\nu_C(z_{ij})}\right) \\ -\left(\pi_A(x_{ij})+\pi_C(z_{ij})\right) \end{array}\right\} \end{array}\right]$$

$$+\frac{-1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c} \left(\frac{\mu_B(y_{ij})+\mu_C(z_{ij})}{2}\right) log \frac{\mu_B(y_{ij})+\mu_C(z_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_C(z_{ij})\right)+\left(\nu_B(y_{ij})+\nu_C(z_{ij})\right)\right)} \\ +\left(\frac{\nu_B(y_{ij})+\nu_C(z_{ij})}{2}\right) log \frac{\nu_B(y_{ij})+\nu_C(z_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_C(z_{ij})\right)+\left(\nu_B(y_{ij})+\nu_C(z_{ij})\right)\right)} \\ -\left(\frac{\pi_B(y_{ij})+\pi_C(z_{ij})}{2}\right) \\ -\frac{1}{2}\left\{\begin{array}{c} \mu_B(y_{ij}) log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_{A\cup B}(y_{ij})}\right) \\ +\nu_B(y_{ij}) log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right) \\ +\mu_C(z_{ij}) log\left(\frac{\mu_C(z_{ij})}{\mu_C(z_{ij})+\nu_C(z_{ij})}\right) \\ +\nu_C(y_{ij}) log\left(\frac{\nu_C(z_{ij})}{\mu_C(z_{ij})+\nu_C(z_{ij})}\right) \\ -\left(\pi_B(y_{ij})+\pi_C(z_{ij})\right) \end{array}\right\} \end{array}\right]$$

$$-\frac{(-1)}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\begin{array}{c} \left(\frac{\mu_{A\cup B}(x_{ij})+\mu_C(z_{ij})}{2}\right) log \frac{\mu_{A\cup B}(x_{ij})+\mu_C(z_{ij})}{\left(\left(\mu_{A\cup B}(x_{ij})+\mu_C(z_{ij})\right)+\left(\nu_{A\cup B}(x_{ij})+\nu_C(z_{ij})\right)\right)} \\ +\left(\frac{\nu_{A\cup B}(x_{ij})+\nu_C(z_{ij})}{2}\right) log \frac{\nu_{A\cup B}(x_{ij})+\nu_C(z_{ij})}{\left(\left(\mu_{A\cup B}(x_{ij})+\mu_C(z_{ij})\right)+\left(\nu_{A\cup B}(x_{ij})+\nu_C(z_{ij})\right)\right)} \\ -\left(\frac{\pi_{A\cup B}(x_{ij})+\pi_C(z_{ij})}{2}\right) \\ -\frac{1}{2}\left\{\begin{array}{c} \mu_{A\cup B}(x_{ij}) log\left(\frac{\mu_{A\cup B}(x_{ij})}{\mu_{A\cup B}(x_{ij})+\nu_{A\cup B}(x_{ij})}\right) \\ +\nu_{A\cup B}(x_{ij}) log\left(\frac{\nu_{A\cup B}(x_{ij})}{\mu_{A\cup B}(x_{ij})+\nu_{A\cup B}(x_{ij})}\right) \\ +\mu_C(z_{ij}) log\left(\frac{\mu_C(z_{ij})}{\mu_C(z_{ij})+\nu_C(z_{ij})}\right) \\ +\nu_C(y_{ij}) log\left(\frac{\nu_C(z_{ij})}{\mu_C(z_{ij})+\nu_C(z_{ij})}\right) \\ -\left(\pi_{A\cup B}(x_{ij})+\pi_C(z_{ij})\right) \end{array}\right\} \end{array}\right]$$

$$=$$

$$\frac{-1}{mn}\sum_{x_{ij},y_{ij}\in G_1}\left[\begin{array}{c} log \frac{\mu_B(y_{ij})+\mu_C(z_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_C(z_{ij})\right)+\left(\nu_B(y_{ij})+\nu_C(z_{ij})\right)\right)} \\ +\left(\frac{\nu_B(y_{ij})+\nu_C(z_{ij})}{2}\right) log \frac{\nu_B(y_{ij})+\nu_C(z_{ij})}{\left(\left(\mu_B(y_{ij})+\mu_C(z_{ij})\right)+\left(\nu_B(y_{ij})+\nu_C(z_{ij})\right)\right)} \\ -\left(\frac{\pi_B(y_{ij})+\pi_C(z_{ij})}{2}\right) \\ -\frac{1}{2}\left\{\begin{array}{c} \mu_B(y_{ij}) log\left(\frac{\mu_B(y_{ij})}{\mu_B(y_{ij})+\nu_{A\cup B}(y_{ij})}\right) \\ +\nu_B(y_{ij}) log\left(\frac{\nu_B(y_{ij})}{\mu_B(y_{ij})+\nu_B(y_{ij})}\right) \\ +\mu_C(z_{ij}) log\left(\frac{\mu_C(z_{ij})}{\mu_C(z_{ij})+\nu_C(z_{ij})}\right) \\ +\nu_C(y_{ij}) log\left(\frac{\nu_C(z_{ij})}{\mu_C(z_{ij})+\nu_C(z_{ij})}\right) \\ -\left(\pi_B(y_{ij})+\pi_C(z_{ij})\right) \end{array}\right\} \end{array}\right]$$

$$+\sum_{x_{ij},y_{ij}\in G_2}\left[\begin{array}{c} \left(\frac{\mu_A(x_{ij})+\mu_C(z_{ij})}{2}\right) log \frac{\mu_A(x_{ij})+\mu_C(z_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_C(z_{ij})\right)+\left(\nu_A(x_{ij})+\nu_C(z_{ij})\right)\right)} \\ -\left(\frac{\pi_A(x_{ij})+\pi_C(z_{ij})}{2}\right) \\ +\left(\frac{\nu_A(x_{ij})+\nu_C(z_{ij})}{2}\right) log \frac{\nu_A(x_{ij})+\nu_C(z_{ij})}{\left(\left(\mu_A(x_{ij})+\mu_C(z_{ij})\right)+\left(\nu_A(x_{ij})+\nu_C(z_{ij})\right)\right)} \\ -\frac{1}{2}\left\{\begin{array}{c} \mu_A(x_{ij}) log\left(\frac{\mu_A(x_{ij})}{\mu_{A\cup B}(x_{ij})+\nu_{A\cup B}(x_{ij})}\right) \\ +\nu_A(x_{ij}) log\left(\frac{\nu_A(x_{ij})}{\mu_A(x_{ij})+\nu_A(x_{ij})}\right) \\ +\mu_C(z_{ij}) log\left(\frac{\mu_C(z_{ij})}{\mu_C(z_{ij})+\nu_C(z_{ij})}\right) \\ +\nu_C(y_{ij}) log\left(\frac{\nu_C(z_{ij})}{\mu_C(z_{ij})+\nu_C(z_{ij})}\right) \\ -\left(\pi_A(x_{ij})+\pi_C(z_{ij})\right) \end{array}\right\} \end{array}\right]$$

$$\geq 0$$

This proves the theorem**.**

Theorem(P8): $Ɗ(A\cap B, C) \leq Ɗ(A, C) + Ɗ(B, C)$.

Proof: We omit proof because theorem as it is similar to theorem (P7).

Theorem (P9): $Ɗ(A \cap C, B \cap C) \leq Ɗ(A, B)$ $for\ all$ $C \in [\ddot{I}F(M)]_{m\times n}$.

Theorem(P10): $Ɗ(A\cup C, B \cup C) \leq Ɗ(A, B) for\ all\ C \in [\ddot{I}F(M)]_{m\times n}$.

Proof: To prove theorem (P9) and (P10), we define eight subsets as follows:

$$S = \{s_i \in S|\ A(s_i) \leq B(s_i) = C(s_i)\} \cup \{s_i \in S|\ A(s_i) = C(s_i) \leq B(s_i)\}$$

$$\cup\{s_i \in S|\ A(s_i) \leq B(s_i) < C(s_i)\} \cup \{s_i \in S|\ A(s_i) \leq C(s_i) < B(s_i)\}$$

$$\cup\{s_i \in S \mid A(s_i) < B(s_i) \le C(s_i)\}\cup\{s_i \in S \mid A(s_i) \le C(s_i) < B(s_i)\}$$

$$\cup\{s_i \in S \mid C(s_i) < A(s_i) \le B(s_i)\}\cup\{s_i \in S \mid C(s_i) < B(s_i) < A(s_i)\}$$

From Montes et al.(2002),

$$|(A\cup C)(s_i) - (B \cup C)(s_i)| \le |A(s_i) - B(s_i)| \; and$$
$$|(A\cap C)(s_i) - (B \cap C)(s_i)| \le |A(s_i) - B(s_i)|$$

Therefore, from Theorem (P5), we get

$\mathcal{D}(A \cap C, B \cap C) \le \mathcal{D}(A, B)$ and $\mathcal{D}(A\cup C, B \cup C) \le \mathcal{D}(A, B)$ *for all* $C \in [\text{IF}(M)]_{m\times n}$.

4.2 Application in medical diagnosis

The divergence measure for IFM can be utilized to quantify the significance of attribute in a specified organized task.
In the study, the proposed divergence measure is applied on a real data taken from a specialist (decision maker). The case study consists of four patients having six symptoms and decision- maker provide information about possibility of disease patient may suffer in intuitionistic fuzzy form. He also provides us Intuitionistic Fuzzy values for symptoms corresponding to disease as well as about patient.
Our main motive is to find closeness of patient and disease based on symptoms. The application of above measure as follows.

4.3 Algorithm for application in multiple-criteria decision-making

IFM is a suitable tool for better demonstrating the imperfectly defined facts and data, as well as imprecise knowledge than fuzzy matrix (FM). In this section, we present a five-stage technique to solve a multiple criteria decision-making problem under an intuitionistic fuzzy environment.

Method:
Let $C = (c_1, c_2, \dots c_m)$ be the set of choices and $F = (f_1, f_2, \dots, f_n)$ be the set of criteria. Assume that the characteristics of the choice C_k in terms criteria F, entered by the decision maker, are represented by the following IFSs:
$C_k = \{\langle F_i, (\mu_{c_k}, \nu_{c_k})\rangle \mid F_i \in F\}$, k = 1, 2,…, m.
Where μ_{c_k} indicates the degree that the C_k (choice) satisfies the criteria F_i and ν_{c_k} indicates the degree that choice C_k does not satisfy the criteria F_i.
Using the divergence measure (DMIFM) defined in eq. (), we set out the following approach to solve multi-decision -making problems.
Step 1: Find the ideal solution I defined as follows:

I = $\{\langle F_i, (\mu_I(F_i), \nu_I(F_i))\rangle \mid F_i \in F\}$ where for each i=1, 2,..., n.

$$(\mu_I(F_i), \nu_I(F_i)) = \left(\max \mu_{c_k}(F_i)\right), \min \mu_{c_k}(F_i))$$

Step 2: Construct an Intuitionistic fuzzy matrix of choices by taking choices as rows and parameters (criteria) as columns
$[\tilde{C}_{ij}] = [c_{ij}]_{m\times n}$
Step 3: Calculate divergence between ideal solution I and in each row matrix of $[c_{ij}]_{1\times n}$ by the proposed measure.

Step 4: Find the minimum value from the divergence which shows the best alternative out of other.
Step 5: Stop.

4.4 A case study

There are four patients of a Doctor Suraj, Raunak, Aditya and Akash in a clinic. They are suffering from a particular disease having some symptoms. The doctor needs to take decision which patient is suffering from which disease. Our proposed measure helps the doctor to take decision for the same. To solve this multiple criterion decision-making problem, we considered some symptoms corresponding to diseases and patients.

The set of symptoms is considered as:

1. Fever
2. Cough
3. Abdominal pain
4. Nausea
5. Chest pain
6. Body ache.

The set of disease is considered as:

1. Malaria
2. Chikungunya
3. Typhoid
4. Stomach infection
5. Respiratory
6. Dengue

Let $P = \{p_1, p_2, p_3, p_4\}$ be the set of patients. The set of symptoms is$\dot{S} = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ and the set of disease is considered as $\dot{D} = \{d_1, d_2, d_3, d_4, d_5, d_6\}$.

Now, the set representation of patient corresponding to symptoms is as follows:

$$P_1 = \{(\dot{S}_1, < [0.8,0.1] >), (\dot{S}_2, < [0.55,0.1] >)(\dot{S}_3, < [0.2,0.8] >), (\dot{S}_4, < [0.6,0.2] >),$$

$$(\dot{S}_5, < [0.1,0.6] >), (\dot{S}_6, < [0.5,0.1] >)\}$$

$$P_2 = \{(\dot{S}_1, < [0.7,0.2] >), (\dot{S}_2, < [0.65,0.10] >)(\dot{S}_3, < [0.8,0.1] >), (\dot{S}_4, < [0.8,0.1] >),$$

$$(\dot{S}_5, < [0.1,0.7] >), (\dot{S}_6, < [0.1,0.7] >)\}$$

$$P_3 = \{(\dot{S}_1, < [0.20,0.6] >), (\dot{S}_2, < [0.8,0.1] >), (\dot{S}_3, < [0.1,0.8] >), (\dot{S}_4, < [0.1,0.8] >),$$

$$(\dot{S}_5, < [0.7,0.1] >), (\dot{S}_6, < [0.1,0.7] >)\}$$

$$P_4 = \{(\dot{S}_1, < [0.7,0.1] >), (\dot{S}_2, < [0.6,0.2] >)(\dot{S}_3, < [0.2,0.7] >), (\dot{S}_4, < [0.2,0.6] >),$$

$$(\dot{S}_5, < [0.2,0.6] >), (\dot{S}_6, < [0.7,0.1] >)\}$$

The tabular representation (Table 1) of these four sets is as follows:

Table 1: Intuitionistic fuzzy values for patients having Symptoms

Patient\symptom	Fever	Cough	Abdominal pain	Nausea	Chest pain	Body ache
P_1	[0.8,0.1]	[0.55,0.1]	[0.2,0.8]	[0.6, 0.2]	[0.1, 0.6]	[0.5, 0.1]
P_2	[0.7,0.2]	[0.65,0.1]	[0.8,0.1]	[0.8, 0.1]	[0.1, 0.7]	[0.1, 0.7]
P_3	[0.2,0.6]	[0.8,0.1]	[0.1,0.8]	[0.1, 0.8]	[0.7, 0.1]	[0.1, 0.7]
P_4	[0.7,0.1]	[0.6,0.2]	[0.2,0.7]	[0.2, 0.6]	[0.2, 0.6]	[0.7, 0.1]

The Matrix Representation of Patient – Symptom is as follows:

$$P = \begin{bmatrix} [0.8,0.1] & [0.55,0.1] & [0.2,0.8] & [0.6,0.2] & [0.1,0.6] & [0.5,0.1] \\ [0.7,0.2] & [0.65,0.1] & [0.8,0.1] & [[0.8,0.1] & [0.1,0.7] & [0.1,0.7] \\ [0.2,0.6] & [0.8,0.1] & [0.1,0.8] & [0.1,0.8] & [0.7,0.1] & [0.1,0.7] \\ [0.7,0.1] & [0.6,0.2] & [0.2,0.7] & [0.2,0.6] & [0.2,0.6] & [0.7,0.1] \end{bmatrix}$$

The Tabular Representation (Table 2) of Disease corresponding to Symptoms is:

Table 2: Intuitionistic fuzzy values for symptoms for disease

Disease \Symptom	S_1	S_2	S_3	S_4	S_5	S_6
$\dot{D}_1$	[0.8,0.1]	[0.6,0.3]	[0.7,0.2]	[0.7,0.3]	[0.1,0.9]	[0.7,0.1]
$\dot{D}_2$	[0.9,0.1]	[0.6,0.2]	[0.1,0.8]	[0.25,0.7]	[0.0,0.9]	[0.8,0.1]
$\dot{D}_3$	[0.9,0.1]	[0.8,0.15]	[0.8,0.05]	[0.85,0.1]	[0.1,0.75]	[0.6,0.2]
$\dot{D}_4$	[0.8,0.2]	[0.7,0.2]	[0.9,0.1]	[0.85,0.1]	[0.1,0.8]	[0.2,0.8]
$\dot{D}_5$	[0.25,0.65]	[0.9,0.1]	[0.15,0.85]	[0.2,0.7]	[0.8,0.1]	[0.1,0.8]
$\dot{D}_6$	[0.85,0.1]	[0.6,0.3]	[0.2,0.75]	[0.25,0.65]	[0.2,0.7]	[0.8,0.1]

Matrix Representation of Disease-Symptom Matrix is as:

$$D = \begin{bmatrix} [0.8,0.1] & [0.6,0.3] & [0.7,0.2] & [0.7,0.3] & [0.1,0.9] & [0.7,0.1] \\ [0.9,0.1] & [0.6,0.2] & [0.1,0.8] & [0.25,0.7] & [0.0,0.9] & [0.8,0.1] \\ [0.9,0.1] & [0.8,0.15] & [0.8,0.05] & [0.85,0.1] & [0.10,0.75] & [0.6,0.2] \\ [0.8,0.2] & [0.7,0.2] & [0.9,0.1] & [0.85,0.1] & [0.1,0.8] & [0.2,0.8] \\ [0.25,0.65] & [0.9,0.1] & [0.15,0.85] & [0.2,0.7] & [0.8,0.1] & [0.1,0.8] \\ [0.85,0.1] & [0.6,0.3] & [0.2,0.75] & [0.25,0.65] & [0.2,0.7] & [0.8,0.1] \end{bmatrix}$$

Now, we find the divergence between these two matrices (P And D) or say patient-symptom matrix and disease – symptom matrix by using the proposed measure eq. (4.1):

Now, we find the divergence between P_1 to each D_i by using proposed measure as:

$I(P_1 : D_1) = 3.539078$, $I(P_1 : D_2) = \mathbf{2.06004}$, $I(P_1 : D_3) = 3.577851$, $I(P_1 : D_4) = 3.037955$, $I(P_1 : D_5) = 2.701628$, $I(P_1 : D_6) = 2.887842$

We find the smallest difference is 2.06004 corresponding to D_2. Thus, we conclude that patient one is suffering from disease D_2. Similarly, we find the divergence for each patient from the considered set of disease.

Divergence for each patient from the considered set of disease are given in the following table 3representing minimum value of divergence which shows patient is suffering from the disease.

Table 3: Patient suffering from disease

Patient \ Disease	Malaria	Chikungunya	Typhoid	Stomach infection	Respiratory	Dengue
Suraj	3.539078	**2.06004**	3.577851	3.037955	2.701628	2.887842
Raunak	3.903724	3.064673	3.637417	**2.07845**	3.006128	3.93541
Aditya	2.717735	2.39147	2.726718	2.379052	**2.336949**	3.003302
Akash	3.69964	2.502108	3.684939	3.733669	**2.221304**	3.28197

From table 3 we can conclude that Suraj is suffering from Chikungunya, Rounak is suffering from stomach infection, Aditya is suffering from respiratory problem, Radhika is suffering from respiratory problem.

Application in MCDM: Pattern Recognition:

Divergence Measure are convenient tool to estimate dissimilarity between two probability functions and are therefore applied in various field.

Example 1: Let there are three patterns A, B, Cof type Intuitionistic Fuzzy sets in X. and the IFSs values for these IFSs are as follows:

$$A = \{(0.3, 0.6), (0.6,0.2), (0.4,0.5)\}$$

$$B = \{(0.2, 0.8), (0.2,0.5), (0.3,0.6)\}$$

$$C = \{(0.1, 0.7), (0.4,0.1), (0.6,0.3)\}$$

$$D = \{(0.4, 0.3), (0.7,0.2), (0.1,0.8)\}$$

Assume that a sample P is given as follows

Pattern $P = \{(0.4, 0.5), (0.6,0.4), (0.6,0.1)\}$

Now to find dissimilarity between patterns to sample pattern, we need to apply divergence measure

$D(A, X) = 2.378125$, $,D(B, X) = 1.903091$, $D(C, X) = \mathbf{1.796831}$, $D(D, X) = 2.003087$

Here, divergence value between pattern C and X is minimum. Therefore, we can say that Pattern C is more like X than others.

5 Conclusion

In this paper, we have recommended an adaptable technique for solving multi-criteria decision-making problem. Using the idea of Mishra et al. (2018) a new divergence measure for intuitionistic fuzzy matrix has developed. Proposed measure is a valid measure as it satisfies all the axioms. Some operations which may be applied on intuitionistic fuzzy matrices are defined. Its fundamental advantage is the capacity to combine the subjective information and attitude character of the decision maker in estimating the interaction of divergence degree. The proposed measure satisfies properties which are valuable in tacking any dynamic issue. The intuitionistic fuzzy divergence measure can be utilized in circumstances relying on the importance of decision makers. It can be utilized in situations where data should be in degree of membership and non-membership values. However, in many real-life situations, semantic factors are utilized to address subjective information. An application of multiple criteria decision-making in medical diagnosis is also demonstrated by taking information from a doctor. This divergence measure in medical diagnosis to take decision which disease based on symptoms the patient follows. We may select the best lternative in preference to other available alternatives in multi-criteria decision-making problem. Using the similar methodology different information measures such as entropy, inclusion, similarity etc. can be derived. Also, further hybridization of intuitionistic fuzzy with vague, soft and rough sets can be done to extend the applicability of the same.

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